Wolfgang Vogel's contributions to intersection theory *

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The theory of multiplicities and intersections played a central role in Wolfgang Vogel's work. His interest in these questions was present in his very early work as well as in his last papers. His best known and most significant contribution is a refined version of Bezout's theorem for improper intersections, in joint work with J. Stückrad. This was proved using a very simple and ingenious argument which nowadays is commonly called the Stückrad-Vogel algorithm. Before explaining this algorithm and some of its applications let me look at the early work of Wolfgang.

When he started in Halle as a student, Ott-Heinrich Keller (1906-1990) had the chair of algebra and geometry there. Keller wrote his dissertation under M. Dehn (Göttingen) on topology. Later on he turned to geometry and worked, for example, on plane algebraic curves. He became especially interested in topological questions concerning algebraic varieties and was well aware of the developments of intersection theory, especially the work of van der Waerden, Weil and Gröbner. Moreover he knew the work of Lefschetz in topology. From the time when van der Waerden proved his principle of conservation of intersection numbers using the topological methods of Lefschetz, it was commonly felt that topological methods were an important tool for the study of algebraic varieties. Keller considered it as one of his aims to connect these two theories, partly in his own work and, as well, through discussions with and suggestions to his students; see the obituary

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written by his former students Wolfgang Vogel and L. Stammler [93-5]. It was mainly through him that Wolfgang was introduced into the field of algebra and geometry. When Wolfgang was a student at Halle, there was a whole group of students and assistants of Keller working in algebra, topology and geometry. Among them were M. Herrmann, O. Krötenheerdt, B. Renschuch, L. Stammler and K. Drechsler. From this group the most important for Wolfgang, as a student and for his early work, were Bodo Renschuch and later on also Manfred Herrmann. With these two he published some of his first papers. In particular, Renschuch was interested, in a purely algebraic setting, in the theory of intersection numbers, Hilbert-Samuel multiplicities and also in computational aspects of ideal theory. In the fall of 1966 Wolfgang spent a longish period in Innsbruck, participating in the Seminar of W. Gröbner who came to exert a strong influence on his mathematical thought.

One of the predominant themes in the mathematical work of Gröbner was a general version of Bezout's theorem. Van der Waerden succeeded in 1928 in giving a proof for arbitrary algebraic varieties meeting properly in projective space, see [14]. He defined intersection multiplicities by moving the varieties into general position and then counting points. This required his so-called specialization principle by which he could prove that his intersection numbers were well defined. With this he was able to make precise the ideas of the Italian school of algebraic geometry (see also the interesting article of Severi [13]). An important observation of van der Waerden was that the Bezout theorem does not hold if one replaces the intersection multiplicity mentioned above by the length multiplicity; an example can be found in the above paper. Of course, nowadays we know by the work of Serre [12] that the length is the right measure for the intersection only if the varieties are Cohen-Macaulay.

The objection of Gröbner to the work of van der Waerden was that the definition of intersection multiplicities was too complicated and that a static definition of intersection numbers is required, instead of the "dynamic" approach of van der Waerden using a deformation argument (see [8, 9]). This criticism was justified since van der Waerden could not give a purely algebraic definition of intersection multiplicities along a component of the intersection. Gröbner gave a version of Bezout's theorem for the intersection of a Cohen-Macaulay variety with a complete intersection and observed that in this case one can take in fact the length multiplicity as a measure for the intersection. More precisely, let $X,\ Y \subseteq \mathbf{P}^n$ be algebraic subschemes meeting properly, i.e.

$\dim X \cap Y = n - \dim X - \dim Y$. Assume moreover that $\mathcal{O}_{X,C}$ is a Cohen-Macaulay ring for all irreducible components C of the intersection and that Y is a complete intersection. Then he showed that the theorem of Bezout holds in its naive form, namely

$$\deg X \deg Y = \deg X \cap Y.$$

Moreover he asked for the underlying reasons why in some cases the length did not give the right intersection number.

Of course, nowadays we have satisfactory answers to these questions, thanks to the work of Weil, Samuel and Serre, see [15, 11, 12]. But I have to mention this since Wolfgang was well aware of these historic developments and since it always played a role in his way of thinking, see [74-1, 82-5]. The problem when the length could be taken as the intersection multiplicity especially occurs in many of his early papers, see [66-2, 67-2, 69-3, 72-3]. In particular, he tried to get an estimate or a description for the difference of measures of multiplicity $\lg A/\mathfrak{q} - e(\mathfrak{q}, A)$ in terms of other ring invariants. Eventually this led to the development of the theory of Buchsbaum rings, see the article by J. Stückrad.

In 1981 he heard through his private contacts with U. Storch about a conference in Oberwolfach on algebraic geometry where Fulton had given a talk. At that time modern intersection theory was in development. The driving forces were Fulton and MacPherson, who were able to define intersection products in a very general setting thus leading to significant generalizations of the classical theory including the Riemann Roch theorem. There were lecture notes of Fulton from a summer school in Cortona held in 1980 which were circulating among the experts and to which also the group in Halle had access. In Oberwolfach, Fulton gave a talk about connectedness and intersections. As an application of the theory he could show a refined Bezout theorem that for arbitrary algebraic varieties $X,\ Y \subseteq \mathbf{P}^n$, the number of irreducible components was bounded by the Bezout number $\deg X \deg Y$. This solved a question of S. Kleiman. Clearly, for a proper intersection this was an immediate consequence of Bezout's theorem whereas in the general case one had to apply the machinery of intersection theory and especially the deformation to the normal cone. Wolfgang gave a talk at the Algebra and Geometry Seminar in Halle about the results of Fulton and proposed to give a more elementary proof of the above inequality.

What was very useful was that just before, in May 1978, he had given a lecture at the University of Maine about the classical version of Bezout's theorem where he obtained a simple and elegant proof using implicitly the join construction, see [80-2]. This paper, as well as almost all his papers, is written up in the language of commutative algebra. Deep in his heart he was always an algebraist. He once explained to me that merely with geometry one would have never detected the role of embedded components.

To explain the essential ideas, let me describe them in a more geometric setup. Let X, Y be subvarieties of $\mathbf{P}^n$. To define the intersection, Weil and Samuel had replaced the intersection $X \cap Y$ by the intersection of $X \times Y$ with the diagonal, say, $\Delta_{\mathbf{P}^n}$ in $\mathbf{P}^n \times \mathbf{P}^n$. The advantage is that the diagonal $\Delta_{\mathbf{P}^n}$ is given locally by n equations, which is just the codimension of $X \cap Y = \Delta_{\mathbf{P}^n} \cap X \times Y$ in $\mathbf{P}^n \times \mathbf{P}^n$ if X and Y meet properly. But there is even a more clever way to do this. Let $\hat{X}, \hat{Y}$ be the cones in the affine space $\mathbf{A}^{n+1}$. Taking now the product $\hat{X} \times \hat{Y}$, the intersection with the diagonal, say, Δ in $\mathbf{A}^{n+1} \times \mathbf{A}^{n+1}$ is *globally* given by the expected number of equations, and these are even linear. Therefore, cutting successively with one hyperplane after the other as it was done e.g. in the work of Gröbner, one obtains Bezout's theorem in its general form. Apparently, this simple but very effective trick was not known by the classical geometers, as is clearly visible for instance in the work of Gröbner. Roughly at the same time and independently, this construction was also used by Fulton and MacPherson to prove their refined Bezout theorem. In the notes of the Cortona summer school Fulton credits this construction to Deligne.

So Wolfgang started to work with Stückrad on the case of an improper intersection. They attempted to assign multiplicities j_C to all components of the intersection (including the improper ones) generalizing the Samuel-Weil multiplicity, and hoped for a version of Bezout's theorem like

$$\deg X \deg Y = \sum j_C \deg C$$

where the sum has to be taken over the irreducible components of the intersection. Soon they discovered that there is in fact a simple and very elegant way to do this. The results were published in [82-1]. The idea is very similar to the well known procedure of computing multiplicities using a parameter system, see also [80-1]. A similar method was already used and studied, in connection with other questions, in the thesis of J. Stückrad. Assume that $\mathfrak{a}$, $\mathfrak{b} \subseteq K[X_0, \ldots X_n]$ are the ideals of X, Y, respectively. They formed the polynomial ring $R := K[X_0, \ldots, X_n, Y_0, \ldots, Y_n]$ and considered there the ideal, say, I generated by all polynomials $f(X_0, \ldots X_n)$, $g(Y_0, \ldots Y_n)$ with $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$. In other words, if $\hat{X}$ and $\hat{Y}$ denote the affine cones of X, Y then R/I is the affine coordinate ring of $V := \hat{X} \times \hat{Y}$. Then they considered generic hyperplanes H_i, $0 \leq i \leq n$, given by equations $\ell_i := \sum u_{ij}(X_j - Y_j)$, whose intersection is the diagonal Δ in the space $\mathbf{A}^n \times \mathbf{A}^n$. To make the construction canonical, they used indeterminates u_{ij} for the coefficients, a trick which was already used by earlier geometers, e.g. by Kronecker, van der Waerden, or Hentzelt and Noether. Then the algorithm started. First they intersected V with H_0 and considered the sum of all irreducible components $\hat{C}$ of the intersection $V \cap H_0$ that are contained in the diagonal. Clearly, $\hat{C}$ is given by a homogeneous ideal and so corresponds to a subvariety C of $X \cap Y$. They assigned to $\hat{C}$ or, equivalently, to C the multiplicity $j(X, Y; C) := \lg \mathcal{O}_{V \cap H_0, \hat{C}}$. Then they replaced V by the closure of $V \cap H_0 \setminus \Delta$, i.e. they removed from $V \cap H_0$ all components that were already contained in Δ, and repeated the procedure by cutting with the second generic hyperplane H_1. Eventually they arrived at a cycle

$$v = \sum j(X, Y; C)[C] + j_0,$$

where j_0 in brief denotes the multiplicity of the origin. Now their version of Bezout's theorem for improper intersections is

$$\deg X \deg Y = \sum j(X, Y; C) \deg C + j_0.$$

In general, one has to take into account not only the irreducible components of the intersection; in fact the sum has to be taken over a whole collection of subvarieties defined over the purely transcendental field extension $K(u_{ij})$ of K. The proof of this refined Bezout theorem is very simple and consists just in observing that the degree of a projective variety $Z \subseteq \mathbf{P}^N$ does not change when cutting with a hyperplane H with $Z \not\subseteq H$. In the case of a proper intersection this reduces to the proof mentioned above of the classical theorem of Bezout. And they also obtained a new proof of the result of Fulton and MacPherson that the number of components of an improper intersection is at most the Bezout number $\deg X \deg Y$.

In many subsequent papers this algorithm was further examined, see [83-1, 85-1, 85-2, 85-3, 87-1, 87-3]. Especially they tried to understand the following questions which Wolfgang asked again and again and on which he, together with several coauthors, published many papers.

1) What is the role of embedded components? For instance, is it true that the Bezout number $\deg X \deg Y$ bounds the number of all components of the intersection, including the embedded ones?

2) Which components can occur? For instance, in the case of self-intersection where $X = Y$, which components of the singular set are in the collection of the above subvarieties?

The key to understanding the second question came when van Gastel in his important thesis [6, 7] compared the above algorithm with the intersection theory of Fulton and MacPherson. He was able to interpret the Stückrad-Vogel cycle in terms of Segre classes. This showed that one could compute such Segre classes without passing first to the normal cone. This is a very useful insight and allows one in many cases to deal much more directly with intersections, using the geometric information of the ambient variety which might be lost when passing to the normal cone. The crucial point in proving this is that the whole intersection algorithm is stable under deformation to the normal cone.

Using his methods, he could in particular connect the distinguished components of the Fulton-MacPherson approach with the collection of Stückrad and Vogel. His interesting result was that the distinguished components are just those components of the Stückrad-Vogel cycle that are K-rational. And moreover, equally important is the fact that he could give a geometrical interpretation of the coefficient j_0. For this, let $J \subseteq \mathbf{P}^{2n+1}$ be the ruled join, i.e.

$$J := \{[x, y] \in \mathbf{P}^{2n+1} | x \in X,\ y \in Y\},$$

so that the affine cone over J is just $\hat{X} \times \hat{Y}$. There is a natural rational map $\pi : J \to \mathbf{P}^n$ given by $[x, y] \mapsto x - y$. The image of this map is the embedded join of X, Y which is usually denoted by XY and which, geometrically speaking, is the union of all lines $\overline{xy}$ with $x \in X$ and $y \in Y$. Then it was shown by van Gastel that j_0 is given by $\deg XY \deg \pi$.

I visited Halle for the first time in 1985 to meet Vogel and the group around him. Through this visit I learned about their approach to improper intersections. Wolfgang asked me in particular about the embedded components of the algorithm. He told me about some positive results which he and Stückrad had obtained at that time. Shortly after leaving Halle, I managed to construct a series of examples which showed that in general not all the embedded components can occur in the cycle. Later on, when I came to Halle for a second time, we began a collaboration. Just before this, van Gastel had visited Halle and given a talk about his results. Wolfgang especially asked me about the possibility of giving a converse to Bezout's theorem. So we started to work on this question. In doing this we could also give simplified proofs of some of the results of van Gastel. We published a part of this work together in a joint paper with van Gastel (see [90-1]) where we proved that the join has minimal dimension if and only if

$$\deg X \deg Y = \sum j(X, Y; C) \deg C.$$

Here the sum is taken over all *irreducible* components of the intersection. This does not give a full converse to Bezout's theorem since there are many examples where the join has minimal dimension without the intersection being proper. Therefore we tried to replace the j-multiplicity by another measure for the intersection number. For instance, for reduced Cohen-Macaulay varieties we obtained a converse of the naive form of Bezout's theorem: *the intersection* $X \cap Y$ *is proper if and only if* $\deg X \deg Y = \deg X \cap Y$, see [93-1]. Moreover, in the arithmetically Cohen-Macaulay case, Wolfgang together with Renschuch gave an elementary proof, without the reducedness assumption. What turned out to be important for these results was the connectedness theorem of Grothendieck [10, Exp. XIII]. The importance of this became especially apparent in the work of F. Zak, who succeeded in proving the Hartshorne conjecture on linear normality; see [16] and especially the excellent article of Fulton and Lazarsfeld [5].

In 1990, shortly after the fall of the Iron Curtain, Jürgen Stückrad, together with Vogel, organized a conference on commutative algebra and algebraic geometry in Eisenach. There I also met Liam O'Carroll, and Vogel proposed that we should write a monograph on intersection theory, especially on the algorithm, its connection with Segre classes and some of its applications. This, as it turned out, was one of the driving forces to continue with our collaboration. When writing up the book we met many problems, and several papers originated in solving these problems and in simplifying proofs of known results. One of our main efforts was devoted to understanding better the distinguished varieties. For this we introduced the so-called limit of join variety $\mathrm{LJoin}_C(X, Y)$ at a subvariety C of the intersection which is a local generalization of the join construction. In the case where C is a point, say, P, $\mathrm{LJoin}_C(X, Y)$ is the union of all lines ℓ which are limits $\ell = \lim \overline{x_\nu y_\nu}$ where $x_\nu \in X$ and $y_\nu \in Y$ are sequences tending to P. The dimension of this limit of join variety is always bounded by $\dim X + \dim Y - \dim C$. Moreover equality holds if and only if C is a distinguished component. Thus we obtained a very geometrical way to describe the rational components of the intersection.

The next question we wanted to understand was which subvarieties of the intersection really occur in the intersection cycle. In other words, when is the limit of join variety of maximal dimension? We met B. Ulrich at a conference in Essen and started to discuss with him our problems and the connection with the work which he had recently done together with Simis and Vasconcelos. It turned out that one could apply their techniques, which were based on the theory of residual intersections of Huneke, Ulrich and many others. Out of this collaborations came the paper [97-2], where we could prove a number of criteria. For instance in the case of a self-intersection, we obtained the result that the limit of join variety has maximal dimension for a complete intersection under some mild additional hypotheses which excluded the case of products.

There are further interesting aspects of the Stückrad-Vogel algorithm which I want to mention. It was already observed by van Gastel that the algorithm gives a new approach in understanding Holme's embedding obstruction in a simpler way. This was further exploited in [3]. Another interesting aspect is that it led to a generalization of the notion of multiplicity to ideals which are not necessarily primary, see [1]. And furthermore I believe that this algorithm should also be useful for enumerative questions; some work about this was begun by van Gastel.

In recent years Wolfgang again became very interested in the length multiplicity, thereby returning to the questions with which he started, namely the comparison of the usual multiplicity and the length. This time, it was the quotient rather than the difference of the two that was examined. In collaboration with Stückrad and Miyazaki he was led to several interesting conjectures concerning these numbers. In other joint papers with Sturmfels, Miyazaki and Trung, he tried again to incorporate the contributions of embedded components into various intersection numbers.

Many of the collaborations with Wolfgang started with questions from him. It was always very pleasant to work with him. For instance, shortly after the fall of the Iron Curtain I visited him in Halle. We decided to go to Rügen for a few days and to work there in the relaxing atmosphere of the seaside. We thought about the converse of the Bezout problem and found a whole series of examples which showed that the join can have minimal dimension without the intersection being proper. He was always full of energy. Even a few month before his death, when Wolfgang visited the Max Planck Institute in Bonn and the University of Bochum, we still worked intensely together to complete our book. At that time our impression was that Wolfgang had recovered his health, and we were very hopeful for him. His old energy seemed to have returned. It came as a surprise to us that only a few weeks after leaving Germany he became ill again. With his death I lost not only a close colleague but also a good friend. I believe that his scientific work, especially in intersection theory, will endure and will remind us of him.

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